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Eulerian velocity reconstruction in ideal atmospheric dynamics using potential vorticity and potential temperature

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Abstract

An approach for the reconstruction of the velocity field in ideal atmospheric dynamics for given density and the two tracers potential vorticity and potential temperature (entropy) is presented. The method is based on the fundamental equations without approximation. The key step is to satisfy the continuity equation by the inclusion of a third Lagrangian tracer χ. This field is determined by closure conditions for density, potential vorticity and by boundary conditions. The reconstruction is, using the exterior calculus, $*(\varrho\tilde{U}) = \tilde{\mathrm{d}}\chi \wedge \tilde{\mathrm{d}}Q \wedge \tilde{\mathrm{d}}\theta$ with the four-dimensional 1-form $\tilde{U}$ based on the velocity components $(1, u, v, w)$, density ϱ, potential vorticity Q and potential temperature θ. In the mean atmospheric flow χ represents the initial longitude of a fluid particle. For stationary flows χ is related to the Bernoulli function. Examples with analytical solutions are presented for a Rossby wave and zonal and rotational shear flow.

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1. Introduction

Atmospheric dynamics is governed by the momentum balance, the first law of thermodynamics and the continuity equation. Adiabatic and frictionless (ideal) flow respects two Lagrangian conserved quantities, the potential temperature θ, which is a function of the entropy s, $\theta = \exp(s/c_p)$, and potential vorticity (PV), Q, which is derived from the velocity, θ, and the density ϱ [1]. In this paper a method for deriving the velocity field from given fields of the density, potential temperature and PV is outlined.

At first sight the need for the reconstruction of the velocity from PV seems to be unrealistic since PV is not measured directly but calculated from observed fields. However, there are two reasons for considering the atmospheric motion from a PV perspective. Firstly, models which are based on the Lagrangian PV-advection are under investigation and promise a reliable alternative to the common Eulerian models [2]. These models require the velocity reconstruction at every time step for the advection. Secondly, interpretation of atmospheric motion in terms of PV and potential temperature has turned out to be useful to understand the complex three-dimensional flow [3–5].

The most simple example of an inversion is an incompressible, two-dimensional flow (u, v) without Coriolis force, where the PV is replaced by the vorticity ζ. The PV inversion leads to a Poisson equation for the stream-function ψ with ζ as source

$$\zeta = \nabla^2\psi, \qquad u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}. \tag{1}$$

In this paper, an approach to derive the velocity field from the space- and time-dependent fields of ϱ, Q and θ is presented. It is an extension of the expression for the stationary incompressible velocity

$$\mathbf{u} = a\nabla Q \times \nabla\theta \tag{2}$$

where $\mathbf{u}$ is on the intersection of the surfaces of the conserved fields Q and θ, $\mathbf{u}\cdot\nabla Q = 0$, $\mathbf{u}\cdot\nabla\theta = 0$, and a is a constant [6]. We assume that PV and potential temperature are the sole tracers (see the investigation on the set of independent Lagrangian fields in [7]). The approach introduces a new field, denoted as the χ-potential, which acts like a stream function together with θ and PV. The χ-potential is a third Lagrangian tracer. In a typical geophysical situation, where θ depends mainly on the initial vertical co-ordinate and PV on the latitude (via the Coriolis force), the χ-potential identifies the initial longitude.

The approach is valid for the Euler equations without any approximation. The dynamic equations itself are not relevant, they enter only through the expression for the potential vorticity. The approach reconstructs the part of the velocity field which contributes to PV, i.e. the vortical flow on θ surfaces. The reconstruction cannot yield gradient flows since these do not contribute to the PV (this is not very restrictive since the major part of the large scale atmospheric flow has shear or vorticity). Approximations can be introduced by particular shapes of the given fields θ, PV and ϱ, and by the appropriate expansion of PV.

The paper is organized as follows. In the next section the equations for ideal atmospheric dynamics and the PV are recapitulated. The reconstruction method is presented in section 3 and the necessary conditions for the χ-potential follow in section 4. Stationary flow is considered in section 5 and the relation of the χ-potential to the Bernoulli function is in section 6. Some examples for simple flow configurations with analytical derivations are in section 7. The outcome of the paper is summarized and discussed in section 8.

2. The dynamical equations and PV

In this section, the basic physical equations for ideal atmospheric dynamics are summarized with particular emphasis on the PV [6]. Ideal atmospheric dynamics is determined by the momentum balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\frac{1}{\varrho}\nabla p - \mathbf{k}g \tag{3}$$

with the velocity $\mathbf{u} = (u, v, w)$, the angular velocity of the earth $\boldsymbol{\Omega}$ leading to the Coriolis force, density ϱ, pressure p, gravity g and the radial vector $\mathbf{k}$. The total derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{4}$$

The continuity equation for the density ϱ requires mass conservation by the flow

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0. \tag{5}$$

Finally, the first law of thermodynamics is written for the potential temperature $\theta = T(p_0/p)^{R/c_p}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = 0 \tag{6}$$

where T is the temperature, R is the gas constant, c_p is the heat capacity for constant pressure and p_0 is a constant pressure.

The conservation law for the potential vorticity Q is derived from the momentum balance, the continuity equation and the first law of thermodynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = 0 \tag{7}$$

$$Q = \frac{1}{\varrho}\nabla\theta \cdot (\nabla \times \mathbf{u} + 2\boldsymbol{\Omega}). \tag{8}$$

This conservation law, which was discovered by Ertel [1], has been recognized later on as the main dynamical equation and a useful diagnostic tool for the large scale flow in the ocean and the atmosphere [3, 4] (see also the review [5] of the work of Kleinschmidt).

Potential vorticity is still under investigation and research considers mathematical properties, observations, and the role of PV in diagnostics and dynamics. The so-called impermeability theorem for PV has been introduced [8], with new expressions for PV [9]. The role of PV in modelling is investigated [2, 10] to extend the two-dimensional vorticity dynamics to stratified three-dimensional flows.

3. Solution using a new tracer χ

The initial step of the approach aims to satisfy the continuity equation. This is considered as a four-dimensional (4D) incompressibility condition for the momentum and is solved by a combination of three stream functions. The stream functions are given by Q, θ and an additional new field denoted as the χ-potential. Formally, time is written as the zeroth component, $x_0 = t$, and the 3D wind vector is extended to the 4D velocity $\mathbf{U} = (U_0, u, v, w)$ with $U_0 = 1$. The continuity equation (5) becomes

$$\frac{\partial}{\partial x_\alpha}(\varrho U_\alpha) = 0 \tag{9}$$

with the implicit sum; all Greek indices are in $0, \ldots, 3$. This equation is satisfied by

$$\varrho U_\alpha = \varepsilon_{\alpha\beta\gamma\mu} \frac{\partial \chi}{\partial x_\beta} \frac{\partial Q}{\partial x_\gamma} \frac{\partial \theta}{\partial x_\mu} \tag{10}$$

where $\varepsilon_{\alpha\beta\gamma\mu}$ is the 4D anti-symmetric permutation symbol. Expression (10) is central in the reconstruction. For given ϱ, Q and θ, χ is unknown and has to be determined by further conditions.

For any velocity component only the derivatives with respect to the other coordinates enter (since $\varepsilon_{\alpha\beta\gamma\mu}$ is anti-symmetric). In particular, if one of the three fields depends only on a single coordinate, then the respective velocity component vanishes. For example, if Q depends only on y, then $v = 0$. This is immediately obvious from the Lagrangian point of view, since particles are fixed to a certain latitude in this case. The reconstruction (10) represents a geometric perspective analogous to (2).

Equation (10) may be written using the Jacobian

$$\varrho U_\alpha = \frac{\partial(x_\alpha(t), \chi, Q, \theta)}{\partial(t, x, y, z)} \tag{11}$$

where $x_\alpha(t)$ is the Lagrangian position of a fluid particle with velocity $u_\alpha = \mathrm{d}x_\alpha(t)/\mathrm{d}t$. This includes the density for $\alpha = 0$, $U_0 = 1$ and $x_0(t) = t$.

Since the material derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t} = U_\alpha \frac{\partial}{\partial x_\alpha} \tag{12}$$

the conservation laws for Q and θ (7), (6) are satisfied implicitly by (10) due to the anti-symmetry of $\varepsilon_{\alpha\beta\gamma\mu}$

$$U_\alpha \frac{\partial Q}{\partial x_\alpha} = 0, \qquad U_\alpha \frac{\partial \theta}{\partial x_\alpha} = 0. \tag{13}$$

An important property of (10) is that χ is also a Lagrangian tracer, $U_\alpha \partial\chi/\partial x_\alpha = 0$, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi = 0. \tag{14}$$

Within the framework of the exterior calculus the reconstruction (10) is written as

$$*\,(\varrho\tilde{U}) = \tilde{\mathrm{d}}\chi \wedge \tilde{\mathrm{d}}Q \wedge \tilde{\mathrm{d}}\theta. \tag{15}$$

Details are given in the appendix.

Although the continuity equation is included without any approximation, the approach is valid in general, since approximations are a property of the data used. Below it will be shown that χ depends on the longitude in the mean climatological flow. Since Q mainly identifies the latitude and θ the altitude of an air parcel, χ is the missing longitude coordinate to span the 3D atmospheric space. Therefore, the tracers χ, Q and θ identify fluid particles. On the other hand, by (10), these tracers reconstruct the density, and the Eulerian velocity field. Thus, this approach relates the Lagrangian to the Eulerian representation of atmospheric dynamics.

For the reconstruction, it is necessary that the gradients of the prescribed fields do not vanish if the velocity is finite. This is provided in most parts of the atmosphere, because stratification leads to a strong dependence of θ on z, whereas PV varies predominantly with the latitude due to the absolute vorticity. Fullmer [11] found that the climatological PV-gradient $\partial Q/\partial y$ shows many zeros in all levels and latitudes, and relates these to cyclogenesis. Therefore, a breakdown of the reconstruction is a possible hint to instability. Furthermore, for a proper identification of the fluid particles, it is necessary that the three fields are independent.

4. Conditions for χ

There are three conditions which determine the χ-potential in (10): χ has to yield the given density, the velocity has to be consistent with the given PV and boundary conditions need to be satisfied.

(i) *Density-closure.* The first condition requires that the prescribed density field ϱ is obtained for $\alpha = 0$ in (10) and $U_0 = 1$

$$\varrho = \varepsilon_{0\beta\gamma\mu} \frac{\partial\chi}{\partial x_\beta}\frac{\partial Q}{\partial x_\gamma}\frac{\partial\theta}{\partial x_\mu} = \frac{\partial(\chi, Q, \theta)}{\partial(x, y, z)} \tag{16}$$

with $x_0(t) = U_0 t = t$ in (11). This shows that the mass element is $\mathrm{d}m = \varrho\,\mathrm{d}V = \mathrm{d}\chi\,\mathrm{d}Q\,\mathrm{d}\theta$ if the coordinates given by χ, Q and θ are orthogonal. This condition requires that χ, Q and θ span the 3D space to uniquely identify the fluid particles. The first-order partial differential equation for χ (16) is linear and can be solved by the method of characteristics.

(ii) *PV-closure*. The result for the velocity in (10), when substituted back in (8), has to yield the prescribed PV field. Here the particular expression for PV enters and possible approximations alter it. This PV closure may be formally written as

$$Q = Q[u_i[\chi, Q, \theta], \varrho, \theta]. \tag{17}$$

This is a linear second-order partial differential equation for χ. Since the general expression is rather lengthy, it is not displayed here. The time derivative is always of first order, even for stationary conditions. For time-dependent fields Q and θ, this equation involves the first-order derivatives of these fields. The integration of this equation requires the full time-dependent fields of ϱ, Q and θ. This renders the approach demanding and an exemplary numerical solution is not attempted here. Instead, simplifications for particular configurations with analytical solutions will be presented below.

(iii) *Boundary conditions*. The velocity field has to satisfy boundaries conditions according to the flow geometry. Typically, the vertical velocity has to vanish at the surface, $w = 0$, in $z = 0$. This condition is most easily fulfilled in (10) if θ depends only on z at the surface, since for w only derivatives of θ with respect to t, x and y are calculated. In the general case of a bottom topography $z = h(x, y)$ we may require a slip boundary condition for the velocity $w = u\partial h/\partial x + v\partial h/\partial y$. For the present approach it is useful that the effects of topography and varying potential temperature on the boundary can be simplified by a constant potential temperature and a compensating surface PV [12].

5. Stationary flow

For a stationary flow the conditions for χ are concise. All prescribed fields are constant, $\partial_t Q = \partial_t\theta = \partial_t\varrho = 0$; however χ depends on time. This case is used to illuminate the reconstruction. Density remains as in (16) and the three-dimensional velocity $\mathbf{u}$ is (10)

$$\varrho\mathbf{u} = -\frac{\partial\chi}{\partial t}\nabla Q \times \nabla\theta. \tag{18}$$

This reflects the common fact that the velocity is parallel to the intersection of the two conserved fields PV and θ. χ is determined by the density and the PV closure.

In a further simplification we require constant density and vertical stratification, $\theta = \theta(z)$. Density is given by

$$\varrho = \left(\frac{\partial\chi}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial\chi}{\partial y}\frac{\partial Q}{\partial x}\right)\frac{\partial\theta}{\partial z} \tag{19}$$

or with the Jacobian, $\varrho = J(\chi, Q)\partial_z\theta$, $J(A, B) = \partial_x A\partial_y B - \partial_y A\partial_x B$. This linear first-order equation determines the spatial dependence of χ.

The velocity is

$$\varrho u = -\frac{\partial\chi}{\partial t}\frac{\partial Q}{\partial y}\frac{\partial\theta}{\partial z}, \qquad \varrho v = \frac{\partial\chi}{\partial t}\frac{\partial Q}{\partial x}\frac{\partial\theta}{\partial z}. \tag{20}$$

The PV is proportional to a stream function. The time dependence of χ is determined by the PV closure (17)

$$Q = \frac{1}{\varrho^2}\left(\frac{\partial\theta}{\partial z}\right)^2\left[\frac{\partial\chi}{\partial t}\left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2}\right) + \frac{\partial^2\chi}{\partial y\partial t}\frac{\partial Q}{\partial y} + \frac{\partial^2\chi}{\partial x\partial t}\frac{\partial Q}{\partial x}\right] + \frac{1}{\varrho}\frac{\partial\theta}{\partial z}f. \tag{21}$$

6. Stationary flow and the Bernoulli function

The Bernoulli function is a measure of the total energy of a fluid particle. For compressible flows, the Bernoulli function is

$$B = h + gz + \mathbf{u}^2/2 \tag{22}$$

with the enthalpy

$$h = c_p T = c_v T + p/\varrho \tag{23}$$

where c_p and c_v are standard heat capacities. The important role of the Bernoulli function originates in the fact that the momentum equation can be written as

$$\frac{\partial}{\partial t}\mathbf{u} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla B + h\nabla\theta/\theta \tag{24}$$

hence for stationary flows, B is constant along the trajectories (stream-lines), $\mathbf{u} \cdot \nabla B = 0$. Note that $\nabla h = \nabla p/\varrho + h\nabla\theta/\theta$ and $T\nabla s = h\nabla\theta/\theta$ with the entropy s.

For stationary flows, B is a function of Q and θ, $B = B(Q, \theta)$. The stationary PV flux without diabatic and friction terms is [13]

$$\varrho\mathbf{u}Q = \nabla\theta \times \nabla B. \tag{25}$$

Using $\varrho\mathbf{u} = -\partial_t \chi \nabla Q \times \nabla\theta$ (18) we can relate the χ-potential to the Bernoulli function

$$\frac{\partial\chi}{\partial t} = \frac{1}{Q}\frac{\partial B}{\partial Q}. \tag{26}$$

This is consistent with the result of Kanehisa [14] with the stream-function ψ

$$\frac{\partial\chi}{\partial t} = \frac{\partial\psi}{\partial Q} \tag{27}$$

for $\varrho\mathbf{u} = -\nabla\psi \times \nabla\theta$ and $Q = \partial B/\partial\psi$.

7. Analytically solved examples

In this section three examples for the reconstruction of the velocity field for given ϱ, Q and θ configurations are presented. The configurations of zonal and axisymmetric flow with shear and a Rossby wave are simple enough to enable an analytical solution. In order to concentrate on different PV fields, the flow is incompressible and purely horizontal in all cases. Although the mathematics is similar for the horizontal shear flow and the axisymmetric flow, these are included since both pertain to different meteorological scales and applications.

7.1. Horizontal shear flow

We consider an example with a vertical potential temperature gradient, $\theta = \theta(z)$, and a constant density ϱ_0. The aim is to reconstruct the flow for a given meridional PV profile, $Q = Q(y)$, with arbitrary dependence on y. Clearly, the flow is incompressible and purely zonal with a meridional shear. It applies to localized jets or regular flow patterns. The ϱ-closure (16), (19) requires

$$\varrho_0 = \frac{\partial\chi}{\partial x}\frac{\partial Q}{\partial y}\frac{\partial\theta}{\partial z} \tag{28}$$

hence

$$\chi(x, y, t) = \frac{\varrho_0 x}{\partial_y Q \partial_z \theta} + \chi_1(y, t) \tag{29}$$

where the first part is time independent and the second part, χ_1, does not depend on x. The PV closure (17), (21) is used to determine $\chi_1(y,t)$, and reads for χ

$$Q = \frac{1}{\varrho_0^2}\left(\frac{\partial\theta}{\partial z}\right)^2 \frac{\partial}{\partial y}\left(\frac{\partial\chi}{\partial t}\frac{\partial Q}{\partial y}\right) + \frac{1}{\varrho_0}\frac{\partial\theta}{\partial z} f(y) \tag{30}$$

with the absolute vorticity $f(y)$. With $\chi_1(y,t) = g(y)t$, the solution of this equation is found for time-independent $g(y)$

$$g(y) = \frac{1}{\partial_y Q}\left(\int^y \frac{\varrho_0}{\partial_z\theta}\left(\frac{\varrho_0}{\partial_z\theta}Q(y') - f(y')\right)\mathrm{d}y' + C_1\right). \tag{31}$$

C_1 represents an arbitrary additive homogeneous solution, which is not relevant because it is cancelled in (10), like any function of Q or of a derivative. $g(y)$ is determined by the relative vorticity part of the PV.

The problem is solved by the χ-potential

$$\chi = \frac{\varrho_0}{\partial_y Q \partial_z\theta}\left(x + \frac{1}{\varrho_0}\frac{\partial Q}{\partial y}\frac{\partial\theta}{\partial z}g(y)t\right). \tag{32}$$

Using this the zonal velocity is derived by $u = -\partial_t\chi\partial_y Q\partial_z\theta/\varrho_0$. The potential χ is proportional to $x - ut$, and hence, to the initial longitude, x_0, of a particle which propagates as $x = x_0 + ut$. Therefore, χ is the Lagrangian identifier for the longitude co-ordinate.

7.2. *Axisymmetric PV distribution*

In this example we consider a given stationary, axisymmetric, but otherwise arbitrary distribution of PV, $Q = Q(r)$, where r is the radial distance. Although this is analogous to the preceding configuration it is included here since it is hydrodynamically relevant. Planetary rotation is neglected here. Density $\varrho_0 = \text{const}$ and potential temperature is stratified, $\theta = \theta(z)$, hence vertical velocity vanishes. For concentrated PV and large distances, this includes point vortices. The aim is to determine the flow (10) using the χ-potential. The radial velocity vanishes, $u_r = 0$, since PV restricts the flow to rotations.

From the ϱ-closure (16) we find for the density

$$\varrho_0 = -\frac{1}{r}\frac{\partial\chi}{\partial\phi}\frac{\partial Q}{\partial r}\frac{\partial\theta}{\partial z} \tag{33}$$

where ϕ is the azimuthal angle. $\partial_\phi\chi$ is stationary. This is solved by

$$\chi(r,\phi,t) = -\frac{r\phi\varrho_0}{\partial_r Q\partial_z\theta} + \chi_1(r,t) \tag{34}$$

where χ_1 is independent of ϕ. The PV closure (17) is used to determine this part, $\chi_1(r,t)$

$$Q = \frac{1}{r\varrho_0^2}\left(\frac{\partial\theta}{\partial z}\right)^2\left[\frac{\partial\chi}{\partial t}\frac{\partial^2 Q}{\partial r^2} + \frac{\partial^2\chi}{\partial r\partial t}\frac{\partial Q}{\partial r}\right]. \tag{35}$$

The solution for $\chi_1 = g(r)t$ is

$$g(r) = \frac{1}{\partial_r Q}\frac{\varrho_0^2}{(\partial_z\theta)^2}\int^r r'Q(r')\,\mathrm{d}r'. \tag{36}$$

With (34) the solution is

$$\chi(r,\phi,t) = -\frac{r\varrho_0}{\partial_r Q\partial_z\theta}(\phi - u_\phi t) \tag{37}$$

where the azimuthal flow is given by

$$u_\phi = \frac{1}{r\varrho_0}\frac{\partial\chi}{\partial t}\frac{\partial Q}{\partial r}\frac{\partial\theta}{\partial z}. \tag{38}$$

In this example, χ is given by the initial azimuth angle. For the decaying PV distribution, $Q = a\partial_z\theta/r\varrho_0$, the angular velocity is constant everywhere, $u_\phi = a$. For the concentrated, bell-shaped vorticity, $Q = (\partial_z\theta/\Gamma\varrho_0)\exp(-r^2/2\Gamma^2)$, we obtain $u_\phi = (\Gamma/r)[1 - \exp(-r^2/2\Gamma^2)]$. For large distances this approaches the rotation $u_\phi = \Gamma/r$ around a point vortex with intensity Γ.

7.3. Rossby wave

Here we consider a Rossby wave on a midlatitude β-channel. Density ϱ_0 is constant, and stratification is $\theta = \theta(z)$. The Rossby wave is represented by a simple wave with amplitude p_0 in PV and a background vorticity

$$Q = \frac{1}{\varrho_0}\frac{\partial\theta}{\partial z}[a\sin(k(x - ct)) + f_0 + \beta y]. \tag{39}$$

The planetary vorticity f has been linearly approximated on the β-plane, valid within a meridional belt. For χ a solution which satisfies the density closure (16) and the PV closure (17) can be found using an ansatz with two unknown parameters A and $\bar{u}$

$$\chi(x, t) = A(x - \bar{u}t). \tag{40}$$

For simplicity, this χ-potential is independent of y. The density closure yields the first parameter, $A = \varrho_0^2\big/(\partial_z\theta)^2\beta$, and the PV closure yields the second parameter $\bar{u} = c + \beta/k^2$, the dispersion relation for Rossby waves. This is obtained although the vorticity equation is not used directly. The velocity is $(u, v) = (\bar{u}, -(a/k)\cos[k(x - ct)])$. This χ-potential shows that the isolines of χ are not necessarily parallel to the flow.

8. Summary and discussion

An approach for the reconstruction of atmospheric flow is presented which uses space- and time-dependent fields of density ϱ, potential vorticity PV and potential temperature θ. The dynamic equations are not used directly; these impact solely the potential vorticity. The basic idea is to consider the time-dependent continuity equation as a condition for zero divergence of momentum in four dimensions (time and space, with unit velocity in time). This is solved by an ansatz for the four-dimensional momentum using three stream functions: the potential vorticiy, potential temperature and a third field, denoted as χ-potential. The reconstruction is inherently time dependent. The χ-potential has to be determined by a density and a potential vorticity closure condition, which are both linear partial differential equations in space and time. In zonal flows, the χ-potential identifies the initial longitude of particles, in addition to potential vorticity and potential temperature which identify mainly meridional and vertical positions. The fields χ, PV and θ determine, on one hand, the Eulerian velocity field, and, on the other hand, are Lagrangian tracers of the fluid. Therefore, the reconstruction combines the Eulerian and the Lagrangian view of hydrodynamics.

The approach requires that the gradients of PV and potential temperature do not vanish when the velocity remains finite. This behaviour indicates a possible, although vague, interrelation with stability conditions. In stationary flows, the χ-potential is related to the Bernoulli function. The approach is outlined for particular flow structures with high symmetry for which analytical solutions can be found.

The application of the reconstruction to approximations of the fundamental equations like the shallow water equations and the primitive equations is straightforward. In the shallow water equations, the potential temperature is replaced by the relative height, $\theta = z/h(x, y, t)$, where h is the height of the fluid. The primitive equations describe a hydrostatic atmosphere where pressure can be used as the vertical coordinate based on $\partial p/\partial z = -g\varrho$. In both examples, a major simplification arises from a constant density.

An obvious problem is that the reconstruction uses time-dependent fields. Therefore, the approach is, at least at the moment, not yet easily applied. Approximations of the fundamental equations modify the expression for the potential vorticity, and these approximations must be consistent with the shape of the given fields. These might be obtained by filtering or averaging of the prescribed fields.

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Appendix. Exterior calculus formulation

The exterior calculus representation is included since it presents the tersest formulation of the present reconstruction [15]. In the four-dimensional Euclidean space (t, x, y, z), the velocity 1-form $\tilde{U}$ is

$$\tilde{U} = U_0\,\tilde{\mathrm{d}}t + u\,\tilde{\mathrm{d}}x + v\,\tilde{\mathrm{d}}y + w\,\tilde{\mathrm{d}}z \tag{A.1}$$

with $U_0 = 1$, and the metric is the diagonal tensor $g_{\alpha\beta} = \delta_{\alpha\beta}$. The metric is required for the mapping of a vector to the corresponding 1-form. The continuity equation (9) is written as divergence

$$\tilde{\mathrm{d}} * (\varrho\tilde{U}) = 0 \tag{A.2}$$

with the exterior derivative $\tilde{\mathrm{d}}$ and the Hodge star operator '$*$'. Here $*(\varrho\tilde{U})$ is a 3-form which corresponds to the 1-form $\varrho\tilde{U}$. The continuity demands that this 3-form is closed.

The advection of a tracer (13), e.g. θ, is given by the scalar product of the two 1-forms

$$(\varrho\tilde{U}, \tilde{\mathrm{d}}\theta) = 0. \tag{A.3}$$

This is equivalent to the Lie-derivative, $L_{\bar{U}}\theta = 0$, with the vector $\bar{U}$ corresponding to the 1-form $\tilde{U}$. On the basis of the definition of the star operator we have the relation

$$(\varrho\tilde{U}, \tilde{\mathrm{d}}\theta)\tilde{\sigma} = -*(\varrho\tilde{U}) \wedge \tilde{\mathrm{d}}\theta \tag{A.4}$$

where $\tilde{\sigma}$ is the volume form in four dimensions

$$\tilde{\sigma} = \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z. \tag{A.5}$$

The reconstructed momentum is given by (10)

$$* (\varrho\tilde{U}) = \tilde{\mathrm{d}}\chi \wedge \tilde{\mathrm{d}}Q \wedge \tilde{\mathrm{d}}\theta \tag{A.6}$$

or

$$\varrho\tilde{U} = -*(\tilde{\mathrm{d}}\chi \wedge \tilde{\mathrm{d}}Q \wedge \tilde{\mathrm{d}}\theta). \tag{A.7}$$

From (A.3) and (A.4) we see that the continuity equation (A.2) is valid for (A.6) since $\tilde{\mathrm{d}}\tilde{\mathrm{d}} = 0$. Furthermore, the three tracers χ, Q and θ are advected by (A.6) since for any function f, $\tilde{\mathrm{d}}f \wedge \tilde{\mathrm{d}}f = 0$.

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